

# Multiple sums and integrals as neutral BKP tau functions<sup>1</sup>

J. Harnad<sup>†b 2</sup>, J.W. van de Leur<sup>\*3</sup> and A. Yu. Orlov<sup>\*\*4</sup>

<sup>†</sup> *Centre de recherches mathématiques, Université de Montréal  
C. P. 6128, succ. centre ville, Montréal, Québec, Canada H3C 3J7*

<sup>b</sup> *Department of Mathematics and Statistics, Concordia University  
7141 Sherbrooke W., Montréal, Québec, Canada H4B 1R6*

<sup>\*</sup> *Mathematical Institute, University of Utrecht,  
P.O. Box 80010, 3508 TA Utrecht, The Netherlands*

<sup>\*\*</sup> *Nonlinear Wave Processes Laboratory,  
Oceanology Institute, 36 Nakhimovskii Prospect  
Moscow 117851, Russia*

## Abstract

We consider multiple sums and multi-integrals as tau functions of the BKP hierarchy using neutral fermions as the simplest tool for deriving these. The sums are over projective Schur functions  $Q_\alpha$  for strict partitions  $\alpha$ . We consider two types of such sums: weighted sums of  $Q_\alpha$  over strict partitions  $\alpha$  and sums over products  $Q_\alpha Q_\gamma$ . In this way we obtain discrete analogues of the beta-ensembles ( $\beta = 1, 2, 4$ ). Continuous versions are represented as multiple integrals. Such sums and integrals are of interest in a number of problems in mathematics and physics.

## 1 Introduction

This work deals with certain multiple sums ((2.1)-(2.7), (2.20)-(2.23)) and multiple integrals ((3.1)-(3.4), (3.20)) that are of interest in a number of problems in mathematics and physics. The multiple sums appear in models of random partitions (as developed by

---

<sup>1</sup>This work was partially supported by the European Union, through the FP6 Marie Curie RTN ENIGMA (Contract no.MRTN-CT-2004-5652) and the European Science Foundation Program MISGAM 8, by the Natural Sciences and Engineering Research Council of Canada (NSERC), the Fonds FCAR du Québec, and by the Russian Academy of Science program “Fundamental Methods in Nonlinear Dynamics”, RFBR-Italian grant No 09-01-92437-KE-a and RFBR grant 08-01-00501.

<sup>2</sup>harnad@crm.umontreal.ca

<sup>3</sup>J.W.vandeLeur@uu.nl

<sup>4</sup>orlovs55@mail.ru

A.Vershik's school; see [1] for a review), and random motion of particles [2] (see also [4] for a review). The multiple integrals yields certain analogues of  $\beta = 1, 2, 4$  ensembles of random matrices, and also an analogue of the two-matrix models [3]. We consider deformations of the measure defined in terms of four semi-infinite sets of parameters and relate the generating function  $Z$  to the coupled two-component BKP hierarchy. The main tool is the use of 1- and 2-component neutral fermions, which provide links with integrable systems known as neutral BKP hierarchies [5], [6]. One version of these was first introduced in [5], [6]. In this work, we use another version introduced in [7].

## 2 Sums over projective Schur functions

In the following, we consider sums over strict partitions, which will be denoted by Greek letters  $\alpha, \beta$ . Recall [8] that a strict partition  $\alpha$  is a set of integers (parts)  $(\alpha_1, \dots, \alpha_k)$  with  $\alpha_1 > \dots > \alpha_k \geq 0$ . The length of a partition  $\alpha$ , denoted  $\ell(\alpha)$ , is the number of nonvanishing parts, thus it is either  $k$  or  $k - 1$ . Let DP be the set of strict partitions (i.e., with distinct parts). We also need a subset of DP, which will be denoted  $DP^2$  and consists of all partitions of the form  $(\alpha_1, \alpha_1 - 1, \alpha_3, \alpha_3 - 1, \dots, \alpha_{k-1}, \alpha_{k-1} - 1)$ . Consider the following sums (for  $L \in \mathbf{N}^+$ ,  $\mathbf{t} := (t_1, t_3, \dots)$ ,  $\mathbf{t}^* := (t_1^*, t_3^*, \dots)$ ,  $\bar{\mathbf{t}} := (\bar{t}_1, \bar{t}_3, \dots)$ ).

$$S_0(\mathbf{t}, L) := \sum_{\substack{\alpha \in \text{DP} \\ \alpha_1 \leq L}} Q_\alpha(\tfrac{1}{2}\mathbf{t}) \quad (2.1)$$

$$S_1(\mathbf{t}, \mathbf{t}^*) := \sum_{\alpha \in \text{DP}} e^{-U_\alpha(\mathbf{t}^*)} Q_\alpha(\tfrac{1}{2}\mathbf{t}) \quad (2.2)$$

$$S_2(\mathbf{t}, \bar{\mathbf{t}}, \mathbf{t}^*) := \sum_{\alpha \in \text{DP}} e^{-U_\alpha(\mathbf{t}^*)} Q_\alpha(\tfrac{1}{2}\mathbf{t}) Q_\alpha(\tfrac{1}{2}\bar{\mathbf{t}}) \quad (2.3)$$

$$S_{00}(\mathbf{t}, \bar{\mathbf{t}}, L) := \sum_{\substack{\alpha \in \text{DP} \\ \alpha_1 \leq L}} Q_\alpha(\tfrac{1}{2}\mathbf{t}) Q_\alpha(\tfrac{1}{2}\bar{\mathbf{t}}) \quad (2.4)$$

$$S_3(\mathbf{t}, A^c) := \sum_{\alpha \in \text{DP}} A_\alpha^c Q_\alpha(\tfrac{1}{2}\mathbf{t}) \quad (2.5)$$

$$S_4(\mathbf{t}, \mathbf{t}^*) := \sum_{\alpha \in \text{DP}^2} e^{-U_\alpha(\mathbf{t}^*)} Q_\alpha(\tfrac{1}{2}\mathbf{t}) \quad (2.6)$$

$$S_5(\mathbf{t}, \bar{\mathbf{t}}, D) := \sum_{\substack{\alpha, \beta \in \text{DP} \\ \ell(\alpha) = \ell(\beta)}} Q_\alpha(\tfrac{1}{2}\mathbf{t}) D_{\alpha, \beta} Q_\beta(\tfrac{1}{2}\bar{\mathbf{t}}) \quad (2.7)$$

Here, the *projective Schur functions*  $Q_\alpha$  are weighted polynomials in the variables  $\mathbf{t} = (t_1, t_3, t_5, \dots)$ ,  $\deg t_m = m$ , labeled by strict partitions (See [8] for their detailed

definition.) Each  $Q_\alpha(\frac{1}{2}\mathbf{t})$  is known to be a BKP tau function [5], [6]. (This was a nice observation of [9], [10]). The fact that only odd subscripts appear in the BKP higher times  $t_{2m-1}$  is related to the reduction from the KP hierarchy. The coefficients  $U_\alpha$  are defined as

$$U_\alpha := \sum_{i=1}^k U_{\alpha_i}, \quad (2.8)$$

where

$$U_n := U_n^{(0)} - \sum_{m \neq 0, \text{odd}} n^m t_m^* - \ln n!, \quad n \in \mathbf{N}^+ \quad (2.9)$$

for some given set of constants  $\{U_n^{(0)}\}$ .

The coefficients  $A_\alpha^c$  on the right hand side of (2.5) are determined in terms a pair  $(A, a)$  where  $A$  is an infinite skew symmetric matrix and  $a$  an infinite vector. For a strict partition  $\alpha = (\alpha_1, \dots, \alpha_k)$ , where  $\alpha_k > 0$ , the numbers  $A_\alpha^c$  are defined as the Pfaffian of an antisymmetric  $2n \times 2n$  matrix  $\tilde{A}$  as follows:

$$A_\alpha^c := \text{Pf}[\tilde{A}] \quad (2.10)$$

where for  $k = 2n$  even

$$\tilde{A}_{ij} = -\tilde{A}_{ji} := A_{\alpha_i, \alpha_j}, \quad 1 \leq i < j \leq 2n \quad (2.11)$$

and for  $k = 2n - 1$  odd

$$\tilde{A}_{ij} = -\tilde{A}_{ji} := \begin{cases} A_{\alpha_i, \alpha_j} & \text{if } 1 \leq i < j \leq 2n - 1 \\ a_{\alpha_i} & \text{if } 1 \leq i < j = 2n. \end{cases} \quad (2.12)$$

In addition we set  $A_0^c = 1$ .

The coefficients  $D_{\alpha, \beta}$  in (2.7) are defined as determinants:

$$D_{\alpha, \beta} = \det (D_{\alpha_i, \beta_j}) \quad (2.13)$$

where  $D$  is a given constant infinite matrix.

**Remark 2.1.** Series (2.1), (2.2), (2.3) and (2.4) may be obtained via specializations of  $A^c$  in the series (2.5). If we put

$$A_{nm} = \frac{1}{2} e^{-U_m - U_n} \text{sgn}(n - m), \quad a_n = e^{-U_n} \quad (2.14)$$

we obtain (2.2). If we further choose  $U_n = 0$  and  $U_n = +\infty$  for  $n \leq L$ ,  $n > L$  respectively, we obtain (2.1). If we set

$$A_{nm} = \frac{1}{2} e^{-U_m - U_n} Q_{(n, m)}(\frac{1}{2}\bar{\mathbf{t}}), \quad a_n = e^{-U_n} Q_{(n)}(\frac{1}{2}\bar{\mathbf{t}}) \quad (2.15)$$

we obtain (2.3). Choosing again  $U_n = 0$ ,  $U_n = +\infty$  for  $n \leq L, n > L$ , we obtain (2.4). The series (2.6) is obtained from (2.5) by taking

$$A_{nm} = \delta_{n+1,m} - \delta_{m+1,n}. \quad (2.16)$$

The sums (2.4) and (2.3) may be also obtained as particular cases of (2.7) by putting

$$D_{nm} = e^{-U_m - U_n} \delta_{n,m} \quad (2.17)$$

to get (2.3).

All these sums are particular examples of BKP tau functions, as introduced in [5], defining solutions to what was called the neutral BKP hierarchy in [7]. They may be further specialized if we choose  $\mathbf{t} = \mathbf{t}_\infty := (1, 0, 0, \dots)$ . Then

$$Q_\alpha(\tfrac{1}{2}\mathbf{t}_\infty) = \Delta^*(\alpha) \prod_{i=1}^k \frac{1}{\alpha_i!}, \quad \alpha = (\alpha_1, \dots, \alpha_k) \quad (2.18)$$

where

$$\Delta^*(\alpha) = \Delta_k^*(\alpha) := \prod_{0 < i < j \leq k} \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \quad (2.19)$$

For this specialization, we have

$$S_1(\mathbf{t}_\infty, \mathbf{t}^*) = \sum_{\alpha \in \text{DP}} \Delta^*(\alpha) \prod_{i=1}^k \frac{e^{-U_{\alpha_i}(\mathbf{t}^*)}}{\alpha_i!} \quad (2.20)$$

$$S_2(\mathbf{t}_\infty, \mathbf{t}_\infty, \mathbf{t}^*) = \sum_{\alpha \in \text{DP}} \Delta^*(\alpha)^2 \prod_{i=1}^k \frac{e^{-U_{\alpha_i}(\mathbf{t}^*)}}{(\alpha_i!)^2} \quad (2.21)$$

$$S_4(\mathbf{t}_\infty, \mathbf{t}^*) = \sum_{\alpha \in \text{DP}'} \tilde{\Delta}^*(\alpha)^4 \prod_{i=1}^k \frac{e^{-U_{\alpha_i}(\mathbf{t}^*) - U_{\alpha_i+1}(\mathbf{t}^*)}}{\alpha_i! (\alpha_i + 1)!} \quad (2.22)$$

$$S_5(\mathbf{t}_\infty, \mathbf{t}_\infty, \mathbf{t}^*) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\substack{\alpha, \beta \in \text{DP} \\ \ell(\alpha) = \ell(\beta) = k}} \Delta^*(\alpha) \Delta^*(\beta) \prod_{i=1}^k \frac{D_{\alpha_i, \beta_i}}{\alpha_i! \beta_i!} \quad (2.23)$$

Here  $\text{DP}'$  is the set of all strict partitions  $(\alpha_1, \alpha_2, \dots, \alpha_N > 0)$  with the property  $\alpha_i > \alpha_{i+1} + 1$ ,  $i = 1, \dots, N-1$ , and

$$\tilde{\Delta}^*(\alpha)^4 := \prod_{i < j \leq N} \frac{(\alpha_i - \alpha_j)^2 ((\alpha_i - \alpha_j)^2 - 1)}{(\alpha_i + \alpha_j)^2 ((\alpha_i + \alpha_j)^2 - 1)}. \quad (2.24)$$

If we replace the term  $\Delta^*$  in the above by the Vandermonde determinant  $\Delta$ , the resulting sums (2.20)-(2.23) may be viewed as discrete analogues of matrix models (see [11]).

**Applications.** The following are some examples of applications of the above sums.

(I) The sum (2.4) was considered first by Tracy and Widom in [12] in a study of the shifted Schur measure.

(II) Sums (2.2) and (2.3) may be viewed as generalizations of hypergeometric functions for the case of many variables. For example, a generalization of the hypergeometric function of type  ${}_pF_r$  may be obtained from expressions (2.2) and (2.3) by defining the parameters  $U_n$  in terms of Gamma functions as follows:

$$U_n = \log \frac{\prod_{i=1}^p \Gamma(n + a_i)}{\prod_{i=1}^r \Gamma(n + b_i)} \quad (2.25)$$

for some set of  $p + r$  constants  $\{a_i, b_i\}$ . Sums of type (2.3) were considered in [13], while sums (2.2) are new. (This case will be considered in detail elsewhere [27].) It may be shown that both series (2.2) and (2.3) may be expressed as Pfaffians of matrices whose entries are expressed via  ${}_pF_r$  and share many properties with the usual hypergeometric functions  ${}_pF_r$ . Analogues of basic hypergeometric functions may be obtained in a similar way.

(III) Consider models of random strict partitions  $\alpha$ , where the relative weight  $W_\alpha$  is given by one of the following:

(A)

$$W_\alpha = A_\alpha^c Q_\alpha(\tfrac{1}{2}\mathbf{t}), \quad (2.26)$$

where  $A^c = (A, a)$  and  $\mathbf{t} = (t_1, t_3, \dots)$  are parameters of the model

(B)

$$W_\alpha = e^{-U_\alpha(\mathbf{t}^*)} Q_\alpha(\tfrac{1}{2}\mathbf{t}), \quad (2.27)$$

where the parameters are  $\mathbf{t} = (t_1, t_3, \dots)$  and  $U = (U_0, U_1, \dots)$ .

(C)

$$W_\alpha = e^{-U_\alpha(\mathbf{t}^*)} Q_\alpha(\tfrac{1}{2}\mathbf{t}) Q_\alpha(\tfrac{1}{2}\bar{\mathbf{t}}), \quad (2.28)$$

where  $\mathbf{t} = (t_1, t_3, \dots)$ ,  $\bar{\mathbf{t}} = (\bar{t}_1, \bar{t}_3, \dots)$  and  $U = (U_0, U_1, \dots)$  are independent parameters. (Note that models (B) and (C) are particular cases of model (A).)

The series  $S_3$ ,  $S_1$  and  $S_2$  may then be viewed as normalization factors (partition functions) respectively for models (A), (B), and (C). Similarly, series  $S_5$  is a partition function for a model of strict bi-partitions.

(IV) Series  $S_1$  and  $S_2$  were used in [18], where random oscillating Young diagrams related to strict partitions were considered.

**Remark 2.2.** The fermionic representation for these models allows their correlation functions to be computed in standard ways. (See e.g. refs. [14], [15], [16] and [17].)

**Neutral fermions.** To construct tau functions of the BKP and two-component BKP hierarchies (see [6], [7]) we need *neutral free fermions*  $\{\phi_i^{(1)}, \phi_i^{(2)}\}_{i \in \mathbf{Z}}$ , satisfying the anti-commutation relations

$$[\phi_n^{(a)}, \phi_m^{(b)}]_+ = (-1)^n \delta_{n,-m} \delta_{a,b}, \quad a, b = 1, 2. \quad (2.29)$$

For one-component BKP only the first component  $\phi_n := \phi_n^{(1)}$ ,  $n \in \mathbb{Z}$  is used. The fermionic Fock space will be chosen as in [7]) (as opposed to [6]). Namely, the action of neutral fermions on vacuum states is defined by

$$\phi_n^{(a)}|0\rangle = 0, \quad \langle 0|\phi_{-n}^{(a)} = 0, \quad n < 0, \quad (2.30)$$

$$\phi_0^{(a)}|0\rangle = \frac{1}{\sqrt{2}}|0\rangle, \quad \langle 0|\phi_0^{(a)} = \frac{1}{\sqrt{2}}\langle 0| \quad (2.31)$$

where relation (2.31) is chosen as in [7]). (See also the Appendix A in the arXiv version of [18] for some details on links between [6] and [7].)

For linear combinations

$$w_k = \sum_a \sum_n c_{k,n}^{(a)} \phi_n^{(a)}, \quad k = 1, 2, \dots, \quad (2.32)$$

Wick's Theorem implies, for arbitrary such products of an even number of  $w_k$ 's

$$\langle w_1 \cdots w_{2n} \rangle = \sum_{\sigma \in S_{2n}} \text{sign}(\sigma) \langle w_{\sigma(1)} w_{\sigma(2)} \rangle \cdots \langle w_{\sigma(2n-1)} w_{\sigma(2n)} \rangle =: \text{Pf} \left( (\langle w_i w_j \rangle)_{1 \leq i, j \leq 2n} \right). \quad (2.33)$$

where the sum is over the permutation group  $S_{2n}$ .

We also need the Fermi fields

$$\phi^{(a)}(z) = \sum_{n \in \mathbb{Z}} \phi_n^{(a)} z^n, \quad a = 1, 2 \quad (2.34)$$

To evaluate integrals we use

$$\langle 0|\phi^{(b)}(z_1)\phi^{(a)}(z_2)|0\rangle = \frac{1}{2} \left( \frac{z_1 - z_2}{z_1 + z_2} \right) \delta_{ab}. \quad (2.35)$$

For  $m \in \mathbf{N}^+$  Wick's theorem implies

$$\langle 0|\phi^{(a)}(z_1)\phi^{(a)}(z_2) \cdots \phi^{(a)}(z_m)|0\rangle = \left( \frac{1}{2} \right)^{\frac{m}{2}} \Delta_m^*(z) \quad (2.36)$$

Note that, because of (2.31), this expectation value is nonvanishing for  $m$  odd.

**BKP tau functions.** The general tau function of the two-component 2-BKP hierarchy is expressed in fermionic form as

$$\begin{aligned} & \tau^{2c-2-BKP}(\mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}, A) \\ &= \langle 0 | \Gamma^{(1)}(\mathbf{t}^{(1)}) \Gamma^{(2)}(\mathbf{t}^{(2)}) e^{\sum_{a,b} \sum_{n,m} A_{nm}^{a,b} \phi_n^{(a)} \phi_m^{(b)}} \bar{\Gamma}^{(1)}(\bar{\mathbf{t}}^{(1)}) \bar{\Gamma}^{(2)}(\bar{\mathbf{t}}^{(2)}) | 0 \rangle. \end{aligned} \quad (2.37)$$

where

$$\Gamma^{(a)}(\mathbf{t}^{(a)}) := \exp \sum_{n \geq 1, \text{ odd}} B_n^{(a)} t_n^{(a)}, \quad \bar{\Gamma}^{(a)}(\bar{\mathbf{t}}^{(a)}) := \exp \sum_{n \geq 1, \text{ odd}} B_{-n}^{(a)} \bar{t}_n^{(a)} \quad (2.38)$$

$$B_n^{(a)} := \frac{1}{2} \sum_{i \in \mathbb{Z}} (-1)^{i+1} \phi_i^{(a)} \phi_{-i-n}^{(a)}. \quad (2.39)$$

The array of numbers  $A = \{A_{nm}^{a,b}, a, b = 1, 2; n, m \in \mathbb{Z}\}$  is the data that determine the two-component 2-BKP tau function. The four sets of independent parameters,  $\mathbf{t}^{(a)} = (t_1^{(a)}, t_3^{(a)}, \dots)$ ,  $\bar{\mathbf{t}}^{(a)} = (\bar{t}_1^{(a)}, \bar{t}_3^{(a)}, \dots)$ , for  $a = 1, 2$ , are called higher times of the hierarchy. If we fix any three sets, the fourth will be the higher times of the usual BKP hierarchy [7]. In this sense (2.37) may be viewed as a four coupled BKP tau function. If we set  $\bar{\mathbf{t}}^{(1)} = \bar{\mathbf{t}}^{(2)} = 0$  we obtain the two-component BKP hierarchy, as described in [7].

For the one-component case we omit the second component and the superscripts. The 2-BKP tau function is then

$$\tau^{2BKP}(\mathbf{t}, \bar{\mathbf{t}}) = \langle 0 | \Gamma(\mathbf{t}) e^{\sum_{n,m \in \mathbb{Z}} A_{nm} \phi_n \phi_m} \Gamma(\bar{\mathbf{t}}) | 0 \rangle \quad (2.40)$$

and the usual neutral BKP tau function may be written

$$\tau^{BKP}(\mathbf{t}) = \langle 0 | \Gamma(\mathbf{t}) e^{\sum_{n,m \in \mathbb{Z}} A_{nm} \phi_n \phi_m} | 0 \rangle \quad (2.41)$$

A remarkable example of a BKP tau function was found in [9]; namely, Schur's  $Q$ -functions  $Q_\alpha$  themselves [8]. These may be expressed fermionically as

$$\langle 0 | \Gamma(\mathbf{t}) \phi_{\alpha_1} \phi_{\alpha_2} \cdots \phi_{\alpha_{2N}} | 0 \rangle = 2^{-\frac{1}{2}\ell(\alpha)} Q_\alpha(\frac{1}{2}\mathbf{t}) \quad (2.42)$$

where  $\alpha_1 > \alpha_2 > \cdots > \alpha_{2N} \geq 0$ . The set  $(\alpha_1, \dots, \alpha_{2N})$  and the partition  $\alpha$  are related as follows: in case  $\alpha_{2N} > 0$ ,  $\alpha = (\alpha_1, \dots, \alpha_{2N})$  and  $\ell(\alpha) = 2N$ , while in case  $\alpha_{2N} = 0$ ,  $\alpha = (\alpha_1, \dots, \alpha_{2N-1})$  and  $\ell(\alpha) = 2N - 1$ .

**Fermionic representation for sums.** The formulae below show that sums (2.1)-(2.6) are BKP tau functions. First, we have

$$S_3(\mathbf{t}, A^c) = \sum_{\alpha \in \text{DP}} A_\alpha^c Q_\alpha(\frac{1}{2}\mathbf{t}) = \langle 0 | \Gamma(\mathbf{t}) e^{2\sum_{n>m>0} A_{nm} \phi_n \phi_m + 2\sum_{n>0} a_n \phi_n \phi_0} | 0 \rangle. \quad (2.43)$$

In view of Remark 2.1 we have fermionic representations for (2.1), (2.2), (2.3), (2.4). In particular

$$S_0(\mathbf{t}, N) = \sum_{\substack{\alpha \in \text{DP} \\ \alpha_1 \leq N}} Q_\alpha(\tfrac{1}{2}\mathbf{t}) = \langle 0 | \Gamma(\mathbf{t}) e^{2\sum_{N \geq n > m \geq 0} \phi_n \phi_m} | 0 \rangle \quad (2.44)$$

$$S_1(\mathbf{t}, \mathbf{t}^*) = \sum_{\alpha \in \text{DP}} e^{-U_\alpha(\mathbf{t}^*)} Q_\alpha(\tfrac{1}{2}\mathbf{t}) = \langle 0 | \Gamma(\mathbf{t}) \mathbb{T}(\mathbf{t}^*) e^{2\sum_{n > m \geq 0} \phi_n \phi_m} | 0 \rangle \quad (2.45)$$

$$S_4(\mathbf{t}, \mathbf{t}^*) = \sum_{\alpha \in \text{DP}^2} e^{-U_\alpha(\mathbf{t}^*)} Q_\alpha(\tfrac{1}{2}\mathbf{t}) = \langle 0 | \Gamma(\mathbf{t}) \mathbb{T}(\mathbf{t}^*) e^{2\sum_{n > 0} \phi_n \phi_{n+1}} | 0 \rangle \quad (2.46)$$

where

$$\mathbb{T}(\mathbf{t}^*) := \exp \left( - \sum_{n > 0} U_n(\mathbf{t}^*) \phi_n \phi_{-n} \right) \quad (2.47)$$

Evaluating at  $\mathbf{t} = \mathbf{t}_\infty$  gives a “solitonic” representation for  $S_1(\mathbf{t} = \mathbf{t}_\infty, \mathbf{t}^*)$  :

$$S_1(\mathbf{t}_\infty, \mathbf{t}^*) = \frac{1}{c} \langle 0 | \Gamma(\mathbf{t}_+^*) e^{2\sum_{n > m > 0} e^{-U_m^{(0)} - U_n^{(0)}} \phi(n) \phi(m) + 2\sum_{n > 0} e^{-U_n^{(0)}} \phi(n) \phi_0} \Gamma(\mathbf{t}_-^*) | 0 \rangle \quad (2.48)$$

where Fermi fields (2.34) are used (see [18]). This shows that this series is a 2-BKP tau function with respect to the higher times  $\mathbf{t}_\pm^*$  defined as  $\mathbf{t}_+^* := (t_1^*, t_3^*, \dots)$  and  $\mathbf{t}_-^* := (-t_{-1}^*, -t_{-3}^*, \dots)$ . Here  $c$  is a normalization factor given by

$$\langle 0 | \Gamma(\mathbf{t}_+^*) \Gamma(\mathbf{t}_-^*) | 0 \rangle = b(\mathbf{t}_-^*, \mathbf{t}_+^*). \quad (2.49)$$

**Remark 2.3.** Another fermionic representation for (2.3),(2.4) was obtained in [13].

Finally, we have

$$S_5(\mathbf{t}^{(1)}, \mathbf{t}^{(2)}, D) = \langle 0 | \Gamma^{(1)}(\mathbf{t}^{(1)}) \Gamma^{(2)}(\mathbf{t}^{(2)}) e^{2\sum_{n, m > 0} D_{nm} \phi_n^{(1)} \phi_m^{(2)}} | 0 \rangle. \quad (2.50)$$

### 3 Multiple integrals

Let  $d\nu$  be a measure supported on a contour  $\gamma$  on the complex plane. We take  $\gamma$  as either of the following two contours:

- (A) An interval on the real axes  $0 \leq z < \infty$
- (B) A segment of the unit circle: given by  $z = e^{i\varphi}$ ,  $0 \leq \varphi \leq \theta$ ,  $0 < \theta < \pi$ ,

Consider the following  $N$ -fold integrals:

$$I_1(N) := \int_\gamma \cdots \int_\gamma |\Delta^*(z)| \prod_{i=1}^N d\nu(z_i) \quad (3.1)$$



$$I_2(N) := \int_{\gamma} \cdots \int_{\gamma} |\Delta^*(z)|^2 \prod_{i=1}^N d\nu(z_i) \quad (3.2)$$

$$I_3(N) := \int_{\gamma} \cdots \int_{\gamma} \Delta^*(z) a^c(\mathbf{z}) \prod_{i=1}^N d\nu(z_i) \quad (3.3)$$

$$I_4(N) := \int_{\gamma} \cdots \int_{\gamma} |\Delta^*(z)|^4 \prod_{i=1}^N d\nu(z_i) \quad (3.4)$$

where, as before,

$$\Delta^*(z) = \prod_{i>j}^N \frac{z_i - z_j}{z_i + z_j}$$

The notation  $a^c(\mathbf{z})$  is analogous to (2.10), denoting the Pfaffian of an antisymmetric matrix  $\tilde{a}$ :

$$a^c(\mathbf{z}) := \text{Pf}[\tilde{a}] \quad (3.5)$$

whose entries are defined, depending on the parity of  $N$ , in terms of a skew symmetric kernel  $a(z, w)$  (possibly, a distribution) and a function (or distribution)  $a(z)$  as follows:

For  $N = 2n$  even

$$\tilde{a}_{ij} = -\tilde{a}_{ji} := a(z_i, z_j), \quad 1 \leq i < j \leq 2n \quad (3.6)$$

For  $N = 2n - 1$  odd

$$\tilde{a}_{ij} = -\tilde{a}_{ji} := \begin{cases} a(z_i, z_j) & \text{if } 1 \leq i < j \leq 2n - 1 \\ a(z_i) & \text{if } 1 \leq i < j = 2n \end{cases} \quad (3.7)$$

In addition we define  $a_0^c = 1$ .

Integrals  $I_1, I_2$  and  $I_4$  may be considered as analogues of  $\beta = 1, 2, 4$  ensembles [3]. They may be obtained as particular cases of  $I_3$  as follows:

Integral  $I_1(N)$  is a particular case of  $I_3(N)$  where in the (A) case

$$a(z_i, z_j) = \text{sgn}(z_i - z_j), \quad a(z) = 1 \quad (3.8)$$

while in case (B)

$$a(z_k, z_j) = e^{-\frac{\pi i}{2}} \text{sgn}(\varphi_k - \varphi_j), \quad a(z) = e^{-\frac{\pi i}{4}}, \quad (3.9)$$

with  $\varphi_i = \arg z_i$ . To prove this we use:

**Lemma 3.1.**

$$\text{Pf} [\text{sgn}(z_k - z_j)] = \text{sgn} \Delta^*(z), \quad z_k \in \mathbb{R}, \quad (3.10)$$

$$\text{Pf} [\text{sgn}(\varphi_k - \varphi_j)] = \text{sgn} \left( e^{-\frac{\pi i}{4}(N^2 - N)} \Delta^*(z) \right), \quad z_k = e^{i\varphi_k} \quad (3.11)$$

where  $k, j = 1, \dots, N$ .

Integral  $I_2(N)$  is obtained from  $I_3(N)$  by setting

$$a(z_i, z_j) = \frac{z_i - z_j}{z_i + z_j}, \quad a(z) = 1. \quad (3.12)$$

We use the fact that

$$\Delta^*(z) = \text{Pf} \left[ \frac{z_i - z_j}{z_i + z_j} \right] \quad (3.13)$$

Integral  $I_4(N)$  is obtained from  $I_3(2N)$  as follows. In case (A) we set

$$a(z_i, z_j) = \frac{1}{2} \left( z_j \frac{\partial}{\partial z_j} \delta(z_i - z_j) - (z_i \leftrightarrow z_j) \right) \quad (3.14)$$

and in case (B) we set

$$a(z_i, z_j) = \frac{\partial}{\partial \varphi_j} \delta(\varphi_i - \varphi_j). \quad (3.15)$$

The integrals containing  $\Delta^*$  may be compared with those defining the partition function of the so-called supersymmetric matrix integrals, see [19]. Integral  $I_2$  defines the partition function of the so-called  $\hat{A}_0$  model (see [20]), the Coulomb gas model with reflection [21], the 1D Ising model [22], and correlation functions in the 2D Ising model [23].

To relate these integrals to the 2-BKP hierarchy we introduce deformations  $I_i(N) \rightarrow I_i(N; \mathbf{t}, \bar{\mathbf{t}})$  through the following deformation of the measure

$$d\nu(z) \rightarrow d\nu(z|\mathbf{t}, \bar{\mathbf{t}}) = b(\mathbf{t}, \{z\})b(-\bar{\mathbf{t}}, \{z^{-1}\})d\nu(z) \quad (3.16)$$

where

$$b(\mathbf{s}, \mathbf{t}) = \exp \sum_{n \text{ odd}} \frac{n}{2} s_n t_n \quad (3.17)$$

and

$$\{z\} = (2z, \frac{2z^3}{3}, \frac{2z^5}{5}, \dots). \quad (3.18)$$

Below, we show that the generating series obtained by Poissonization (the grand partition function)

$$Z_i(\mu; \mathbf{t}, \bar{\mathbf{t}}) = b(\mathbf{t}, \bar{\mathbf{t}}) \sum_{N=0}^{\infty} I_i(N; \mathbf{t}, \bar{\mathbf{t}}) \frac{\mu^N}{N!}, \quad i = 1, 2, 3, 4, \quad (3.19)$$

are particular 2-BKP tau functions (2.40).

We also consider the following  $2N$ -fold integrals:

$$I_5(N; \mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}) := \int \Delta_N^*(z) \Delta_N^*(y) \prod_{i=1}^N d\nu(z_i, y_i | \mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}), \quad (3.20)$$

where

$$d\nu(z, y | \mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}) = b(\mathbf{t}^{(1)}, \{z\}) b(-\bar{\mathbf{t}}^{(1)}, \{z^{-1}\}) b(\mathbf{t}^{(2)}, \{y\}) b(-\bar{\mathbf{t}}^{(2)}, \{y^{-1}\}) d\nu(z, y) \quad (3.21)$$

(here  $d\nu(z, y)$  is an arbitrary bi-measure), and show that the generating series

$$Z_5(\mu; \mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}) = b(\mathbf{t}^{(1)}, \bar{\mathbf{t}}^{(1)}) b(\mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(2)}) \sum_{N=0}^{\infty} I_5(N; \mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}) \frac{\mu^N}{N!} \quad (3.22)$$

is a particular case of the two-component 2-BKP tau function (2.37).

**Remark 3.1.** Note that

$$Z_2(\mu; \mathbf{t}, \bar{\mathbf{t}}) = Z_5(\mu; \mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}) \quad (3.23)$$

if

$$d\nu(z, y) = \delta(z - y) d\nu(z) d\nu(y), \quad \mathbf{t} = \mathbf{t}^{(1)} + \mathbf{t}^{(2)}, \quad \bar{\mathbf{t}} = \bar{\mathbf{t}}^{(1)} + \bar{\mathbf{t}}^{(2)}. \quad (3.24)$$

The integrals  $Z_1(\mu; \mathbf{t}, \bar{\mathbf{t}})$ ,  $Z_2(\mu; \mathbf{t}, \bar{\mathbf{t}})$ ,  $Z_4(\mu; \mathbf{t}, \bar{\mathbf{t}})$  and  $Z_5(\mu; \mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)})$  may be obtained as continuous limits of  $S_1(\mathbf{t}_\infty, \mathbf{t}^*)$ ,  $S_2(\mathbf{t}_\infty, \mathbf{t}_\infty, \mathbf{t}^*)$ ,  $S_4(\mathbf{t}_\infty, \mathbf{t}^*)$  and  $S_5(\mathbf{t}_\infty, \mathbf{t}_\infty, \mathbf{t}^*)$ , respectively.

**Fermionic representation of the integrals.** To obtain the fermionic representation for the integrals above we apply (2.36). Expanding the exponentials and applying Wick's theorem to each term in the sum gives

$$Z_3(\mu; \mathbf{t}, \bar{\mathbf{t}}) = \langle 0 | \Gamma(\mathbf{t}) e^{\mu^2 \int_\gamma \int_\gamma a(z, y) \phi(z) \phi(y) d\nu(z) d\nu(y)} e^{2\mu \int_\gamma a(z) \phi(z) \phi_0 d\nu(z)} \bar{\Gamma}(\bar{\mathbf{t}}) | 0 \rangle. \quad (3.25)$$

Note that

$$Z_3(0; \mathbf{t}, \bar{\mathbf{t}}) = b(\mathbf{t}, \bar{\mathbf{t}}) = \langle 0 | \Gamma(\mathbf{t}) \bar{\Gamma}(\bar{\mathbf{t}}) | 0 \rangle \quad (3.26)$$

was written above in (3.17). In particular we obtain

$$Z_1(\mu; \mathbf{t}, \bar{\mathbf{t}}) = \langle 0 | \Gamma(\mathbf{t}) e^{\mu^2 q^2 \int_\gamma \int_\gamma \text{sgn}(\varsigma(z) - \varsigma(y)) \phi(z) \phi(y) d\nu(z) d\nu(y)} e^{2\mu q \int_\gamma \phi(z) \phi_0 d\nu(z)} \bar{\Gamma}(\bar{\mathbf{t}}) | 0 \rangle \quad (3.27)$$

where  $q = 1$  and  $\varsigma = \varsigma(z)$ ,  $z \in \gamma$  is a parameter on  $\gamma$ , which is equal to  $\varsigma(z) = z$  when the integration contour is  $\mathbb{R}_+$ , while in the case  $z = e^{i\varphi} \in \gamma$ ,  $0 \leq \varphi \leq \theta$ , we set  $\varsigma(z) := \varphi$  and  $q = e^{-\frac{\pi i}{4}}$ .

Similarly, we have

$$Z_2(\mu; \mathbf{t}, \bar{\mathbf{t}}) = \langle 0 | \Gamma(\mathbf{t}) e^{\mu^2 \int_{\gamma} \int_{\gamma} \frac{z-y}{z+y} \phi(z) \phi(y) d\nu(z) d\nu(y)} e^{2\mu \int_{\gamma} \phi(z) \phi_0 d\nu(z)} \bar{\Gamma}(\bar{\mathbf{t}}) | 0 \rangle \quad (3.28)$$

As a specialization of (3.25), by choosing  $a(z, w)$  as in (3.14) and  $a(z) = 0$ , we obtain

$$Z_4(\mu \mathbf{t}, \bar{\mathbf{t}}) = \langle 0 | \Gamma(\mathbf{t}) e^{4\mu \int_{\gamma} z \frac{\partial \phi(z)}{\partial z} \phi(z) d\nu(z)} \bar{\Gamma}(\bar{\mathbf{t}}) | 0 \rangle. \quad (3.29)$$

Finally, in terms of two-component fermions we have

$$\begin{aligned} & Z_5(\mu; \mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}) \\ &= \frac{1}{c} \langle 0 | \Gamma^{(1)}(\mathbf{t}^{(1)}) \Gamma^{(2)}(\mathbf{t}^{(2)}) e^{2\mu \int \int \phi^{(1)}(z) \phi^{(2)}(y) d\nu(z, y)} \bar{\Gamma}^{(1)}(\bar{\mathbf{t}}^{(1)}) \bar{\Gamma}^{(2)}(\bar{\mathbf{t}}^{(2)}) | 0 \rangle, \end{aligned} \quad (3.30)$$

where  $c := Z_5(0; \mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)})$  is the normalization factor.

Formulae (3.25), (3.27), (3.29), and (3.30) should be compared with (2.43), (2.45), (2.46), and (2.50), respectively. The fermionic representations of integrals  $Z_1$  and  $Z_4$  are similar to the results of [24] for ensembles of real symmetric and self-dual quaternionic random matrices.

## 4 Discussion

We have presented five types of multiple sums and multiple integrals which are related to tau functions of BKP type hierarchies (BKP and two-component BKP tau functions for multiple sums, 2-BKP and two-component 2-BKP tau functions for multiple integrals). Certain of these sums and integrals have known applications in mathematics and physics; we believe that all of them may prove to be of use in various probabilistic models. The techniques of free fermion calculus and integrable systems may be applied to study various properties of these sums and integrals. Multicomponent BKP tau functions [7] may be further used to construct models of Pfaffian processes (cf. [26]). The results of this work should be compared with analogous results for the so-called charged BKP case (see [7]) studied in [24] and [27].

## 5 Appendix. An application of the integral $Z_2(N)$ (Harry Braden).

Some of the considerations of this paper were motivated by the following observation of Harry Braden ([23]).

“The application of the integrals  $Z_2$  I have in mind deals with an Ising model correlation function when there is a thermal perturbation from critical temperature. In the scaling limit this is described by a system of Majorana fermions and makes connection with the fermionic representation under consideration. McCoy, Tracy and Wu showed this limit was described a massive ( $m = T - T_c$ ) field theory whose correlations were governed by a radial Sinh-Gordon equation that under a change of parameters is  $P_{III}$ . Another approach to the same correlator is via form factors. This approach yields the correlator in terms of an infinite sum of multiple integrals. For the case at hand this gives a Euclidean correlation function like

$$\begin{aligned} G(r) &:= \langle \mathcal{O}(x) \mathcal{O}(0) \rangle = \sum_{n=0}^{\infty} \int \frac{\prod_{i=1}^n d\beta_i}{n! (2\pi)^n} \langle 0 | \mathcal{O}(x) | \beta_1 \dots \beta_n \rangle \langle \beta_n \dots \beta_1 | \mathcal{O}(0) | 0 \rangle \\ &= \sum_{n=0}^{\infty} \int \frac{\prod_{i=1}^n d\beta_i}{n! (2\pi)^n} |F_n(\beta_1 \dots \beta_n)|^2 \exp(-mr \sum_{i=1}^n \cosh \beta_i). \end{aligned}$$

Here  $\beta_i$  are rapidities and  $x = (x_0, x_1)$ ,  $r = \sqrt{x_0^2 + x_1^2}$ . If we use the minimal form factor

$$F_n^{\min}(\beta_1 \dots \beta_n) = \prod_{i < j} \tanh \left( \frac{\beta_i - \beta_j}{2} \right)$$

and appropriate normalizations we get

$$G\left(\frac{r}{m}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(2\pi)^n} \int_0^{\infty} \prod_{i=1}^n \left( \frac{dx_i}{x_i} e^{-r(x_i + 1/x_i)} \right) \prod_{i < j} \left( \frac{x_i - x_j}{x_i + x_j} \right)^2.$$

As I havent been specific about the precise correlator, we also are interested in

$$G_{\pm}\left(\frac{r}{m}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(\pm 1)^n}{(2\pi)^n} \int_0^{\infty} \prod_{i=1}^n \left( \frac{dx_i}{x_i} e^{-r(x_i + 1/x_i)} \right) \prod_{i < j} \left( \frac{x_i - x_j}{x_i + x_j} \right)^2. \quad (5.1)$$

The identity

$$\det \left( \frac{1}{x_i + x_j} \right) = \frac{1}{2^n x_1 \dots x_n} \prod_{i < j} \left( \frac{x_i - x_j}{x_i + x_j} \right)^2$$

provides a connection with a Fredholm determinant. I have looked at expansions of (5.1). The difficulty is getting a convergent expansion. The first term is in terms of  $K_0(r)$ . Such an expansion will provide one for (a particular) solution of the Painlevé equation.”

## Acknowledgements

The authors are grateful to T. Shiota and J. J. C. Nimmo for useful discussions. One of the authors (A.O.) thanks A. Odziejewicz for kind hospitality during his stay in Białystok in June 2005. Both A.O and J.vdL. thank the CRM, Montréal, Canada for the kind hospitality during their stay in January 2006, where the main part of this paper was written.

## References

- [1] A. Okounkov, "The uses of random partitions", Poceedings of the XIVth International Congress on Mathematics Physics (Lisbon, 2003), World Scientific eproceedings, pp. 379-403. (arXiv:math-ph/0309015)
- [2] M. Fisher, "Walks, walls, wetting and melting", J. Stat. Phys. **34** 667-729 (1984).
- [3] Mehta, M. L., *Random Matrices*, 3nd edition (Elsevier, Academic, San Diego CA, 2004).
- [4] P. J. Forrester, *Log-gases and random matrices*, Princeton University Press, Princeton NJ, (2010).
- [5] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Transformation groups for soliton equations, IV A new hierarchy of soliton equations of KP-type, Physica 4D 343-365 (1982).
- [6] Jimbo, M and Miwa, T,,: Solitons and Infinite Dimensional Lie Algebras, *Publ. RIMS Kyoto Univ.* **19**, 943–1001 (1983)
- [7] V. G. Kac and J. van de Leur, "The geometry of spinors and the multicomponent BKP and DKP", in: *CMR Proceedings and Lecture Notes Vol14*, eds. J. Harnad and A. Kasman (1998).
- [8] I.G.Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edition, Ch. III, Sec. 8, Clarendon Press, Oxford (1995).
- [9] Y. You, "Polynomial solutions of the BKP hierarchy and projective representations of symmetric groups", in *Infinite-dimensional Lie algebras and groups*, Adv. Ser. Math. Phys., **7**, 449–464 (1990). World Science Publishing, Teaneck, New Jersey.
- [10] J. J. C. Nimmo, "Hall-Littlewood symmetric functions and the BKP equation", J. Phys. A, **23**, 751-760 (1990).

- [11] A. Yu. Orlov and T. Shiota, “Schur function expansion for normal matrix model and associated discrete matrix models”, Phys. Lett. A **343**, 384-396 (2005). (For a more complete version, see arXiv:math-ph/0501017)
- [12] C. Tracy and H. Widom, “A Limit Theorem for Shifted Schur Measure”, Duke Math. J. **123**, 171-208 (2004). (arXiv: math.PR/0210255)
- [13] A. Yu. Orlov, ”Hypergeometric functions related to Schur Q-polynomials and BKP equation”, Theoretical and Mathematical Physics, **137** (2): 1573-1588 (2003). (arXiv: math-ph/0302011)
- [14] A. Okounkov, “ $SL(2)$  and  $z$ -measures”, in: *Random Matrix Models and Their Applications*, MSRI publications **40** 407-420 (2001). (arXiv: math.RT/0002135)
- [15] O. Foda, M. Wheeler, M. Zuparic, ”On free fermions and plane partitions”, Journal of Algebra **321** 3249–3273 (2009). (arXiv:hep-th/08082737)
- [16] M. Vuletic, “Shifted Schur Process and Asymptotics of Large Random Strict Plane Partitions”, Int. Math. Res. Notices 2007, Vol **2007**, article ID rnm043, 53 pages. (arXiv:math-ph/0702068v1)
- [17] J. Harnad and A. Yu. Orlov, “Fermionic approach to the evaluation of integrals of rational symmetric functions”, Theor. Math.Phys, **158**, no. 1, 17-39 (2009). (arXiv: math-ph/0704.1150)
- [18] J. van de Leur and A. Yu. Orlov, “Random turn walk on a half line with creation of particles at the origin”, Phys. Lett. A **31**, 2675-2681 (2009). (arxiv: math-ph/08010066)
- [19] T. Guhr, “Dyson’s correlation functions and graded symmetry”, J. Math. Phys. **32**, 336-347 (1991)
- [20] I. K. Kostov, ”Solvable statistical models on a random lattice”, Nucl. Phys. Proc. Suppl. **45A** 13-28 (1996). ( arXiv: hep-th/9509124 v4)
- [21] I. M. Loutsenko and V. P. Spiridonov , “Soliton solutions of integrable hierarchies and Coulomb plasmas”, J. Stat. Phys. **99** , 751-767 (2000). (arXiv: cond-mat/9909308)
- [22] I. M. Loutsenko, and V. P. Spiridonov, ”A Critical Phenomenon in Solitonic Ising Chains”, SIGMA **3** (2007), 059, p. 11. (arXiv: cond-mat/07043173)
- [23] Harry Braden (private communication, May 2006).

- [24] J. van de Leur, "Matrix Integrals and the Geometry of Spinors", J. Nonlinear Math. Phys. ,**8**, no. 2 288-311 (2001). (arXiv: solv-int/9909028 v2)
- [25] J. Harnad and A. Yu. Orlov, "Fermionic construction of partition functions for two-matrix models and perturbative Schur function expansions", J. Phys. **A39** (2006) 8783-8810. (arXiv: math-ph/0512056)
- [26] J. Harnad and A. Yu. Orlov, "Fermionic construction of the partition functions for multi-matrix models and the multi-component TL hierarchy", Theor. Math. Phys. **152**, 1099-1110 (2007). (arXiv: math-ph/07041145)
- [27] A. Yu. Orlov, "Pfaffian structures and certain solutions to DKP and BKP hierarchies", (in preparation).